# Proofs for the general theory of classical electrodynamics

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**Abstract**

Mathematical proofs are given for the majority of equations relating to the general theory of classical electrodynamics. These include equations for the static magnetic vector and scalar fields, the Lorenz condition, power- and force densities, magnetostatic power- and force densities and far field expressions like Jefimenko's expressions. In addition, vector calculus identities are provided as a reference.

## 1 Introduction

This work started as a personal exercise in proofing the main results of [1] which we will refer to here as GCED theory. GCED theory is a generalization of Maxwell's classical electrodynamics which includes the addition of two *scalar fields* and two additional induction equations. GCED theory shows that Lorentz's force law violates Newton's third law of motion for *open circuit* charge-current distributions and resolves this by means of a scalar field. This result is used to restate Maxwell's original classical electrodynamics in a simplified fashion and give a more general form of Maxwell's equations. General power- and force theorems are stated in terms of decomposed and superimposed fields to provide closure to this generalization. Furthermore, GCED theory solves an inconsistency in Jefimenko's equations involving two unexplained far field terms, which predict the existence of two additional waves, namely the *longitudinal electromagnetic wave* and the *'electroscalar' wave* by assuming a superluminal retardation for the electric potential.

The historic significance of GCED theory is realistic in the sense that it is commensurable with the historic and modern *experimental paradigm* of longitudinal waves and superluminal velocities, see [1] paragraph 4 'Review of CED experiments'. Therefore, it is paramount that exact and detailed mathematical proofs of the theoretical results are provided for academic reference. The main results of GCED theory will be restated in the form of formal proofs. Each proof starts by stating the original equation in [1] in a box, using the exact same notation. Then a derivation is given always ending with the original equation. A table with an overview of all theorems including original equation numbers from [1] can be found in appendix D. We refer to the original equations from the GCED paper by prefixing an equation number like this: **G**[x].

In the next paragraph we state the fundamental definitions of GCED theory. These definitions are complemented with basic mathematical notations in appendix A. Useful mathematical identities are provided in appendix B. Detailed proofs for important mathematical identities are given in appendix C.

## 1.1 The foundations of GCED theory

GCED theory is developed by starting with the definitions for the sources, potentials and fields. These equations are used frequently to prove our theorems. The two sources are the charge density and the current density:

$$\text{Charge density} \quad \rho \quad \text{Unit:}\left[\frac{C}{m^3}\right] \tag{1}$$

$$\text{Current density} \quad \mathbf{J} \quad \text{Unit:}\left[\frac{A}{m^2}\right] \tag{2}$$

The two potentials are classically referred to as the electric potential and the magnetic potential. Better descriptions would be the charge potential and current potential. We define the potentials in terms of the sources. The time-independent potentials are:

$$\text{Charge potential} \quad \Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int\limits_{V'} \frac{\rho(\mathbf{x}')}{r}\,\mathrm{d}^3x' \quad \text{Unit:}\Big[V\Big] \tag{3}$$

$$\text{Current potential} \quad \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int\limits_{V'} \frac{\mathbf{J}(\mathbf{x}')}{r}\,\mathrm{d}^3x' \quad \text{Unit:}\left[\frac{kg\cdot m}{c\cdot s}\right] \tag{4}$$

We define the fields in terms of potentials. Besides the two well known electric and magnetic field, GCED theory also models two scalar fields, namely the magnetic scalar field and the 'field induced' scalar field. Furthermore, the electric field is split into three unique parts. See appendix A for notation conventions. Altogether, we define six fields in terms of the potentials:

$$\text{Primary electric field} \quad \mathbf{E}_\Phi = -\nabla\Phi \quad \text{Unit:}\left[\frac{V}{m}\right] \tag{5}$$

$$\text{Field induced divergent electric field} \quad \mathbf{E}_L = -\frac{\partial \mathbf{A}_l}{\partial t} \quad \text{Unit:}\left[\frac{V}{m}\right] \tag{6}$$

$$\text{Field induced rotational electric field} \quad \mathbf{E}_T = -\frac{\partial \mathbf{A}_t}{\partial t} \quad \text{Unit:}\left[\frac{V}{m}\right] \tag{7}$$

$$\text{(Field induced) 'electroscalar' field} \quad B_\Phi = -\frac{\partial \Phi}{\partial t} \quad \text{Unit:}\left[\frac{V}{s}\right] \tag{8}$$

$$\text{Magnetic scalar field} \quad B_L = -\nabla\cdot\mathbf{A}_l \quad \text{Unit:}\left[\frac{V\cdot s}{m^2}\right] \tag{9}$$

$$\text{Magnetic vector field} \quad \mathbf{B}_T = \nabla\times\mathbf{A}_t \quad \text{Unit:}\left[\frac{V\cdot s}{m^2}\right] \tag{10}$$

From the field definitions we can define the following superimposed fields

$$\mathbf{E} = \mathbf{E}_\Phi + \mathbf{E}_L + \mathbf{E}_T \tag{11}$$

$$\mathbf{E}^* = \frac{1}{c^2}\mathbf{E}_\Phi + \frac{1}{b^2}\mathbf{E}_L + \frac{1}{c^2}\mathbf{E}_T \tag{12}$$

$$B = \frac{1}{c^2}B_\Phi + B_L \tag{13}$$

$$B^* = \frac{1}{a^2}B_\Phi + B_L \tag{14}$$

where $c$ is the speed of light and $a$ is an unknown velocity.

GCED theory generalizes Maxwell's source-field and source-potential equations as follows:

$$\nabla\cdot\mathbf{E}_\Phi - \frac{\phi_0}{\epsilon_0}\frac{\partial B_\Phi}{\partial t} = \frac{\phi_0}{\epsilon_0}\frac{\partial^2\Phi}{\partial t^2} - \nabla\cdot\nabla\Phi = \frac{1}{\epsilon_0}\rho \tag{15}$$

$$\nabla B_L - \lambda_0\mu_0\frac{\partial\mathbf{E}_L}{\partial t} = \lambda_0\mu_0\frac{\partial^2\mathbf{A}_l}{\partial t^2} - \nabla\nabla\cdot\mathbf{A}_l = \mu_0\mathbf{J}_l \tag{16}$$

$$\nabla\times\mathbf{B}_T - \epsilon_0\mu_0\frac{\partial\mathbf{E}_T}{\partial t} = \epsilon_0\mu_0\frac{\partial^2\mathbf{A}_t}{\partial t^2} - \nabla\times\nabla\times\mathbf{A}_t = \mu_0\mathbf{J}_t \tag{17}$$

Equation 15 is known as Gauss' law and is generalized by adding the $-\frac{\phi_0}{\epsilon_0}\frac{\partial B_\Phi}{\partial t}$ term. Equations 16 and 17 are a decomposed general form of Ampere's law. Equation 16 includes the $\nabla B_L$ term and the law is decomposed into a curl-free $\mathbf{J}_l$ and a divergence-free $\mathbf{J}_t$ current density source.

Furthermore, Faraday's law is complemented with two additional induction laws stated in 18 and 19

$$\nabla B_\Phi - \frac{\partial\mathbf{E}_\Phi}{\partial t} = -\nabla\frac{\partial\Phi}{\partial t} + \frac{\partial(\nabla\Phi)}{\partial t} = \mathbf{0} \tag{18}$$

$$\nabla\cdot\mathbf{E}_L - \frac{\partial B_L}{\partial t} = -\nabla\cdot\frac{\partial\mathbf{A}_l}{\partial t} + \frac{\partial(\nabla\cdot\mathbf{A}_l)}{\partial t} = 0 \tag{19}$$

$$\nabla\times\mathbf{E}_T + \frac{\partial\mathbf{B}_T}{\partial t} = -\nabla\times\frac{\partial\mathbf{A}_t}{\partial t} + \frac{\partial(\nabla\times\mathbf{A}_t)}{\partial t} = \mathbf{0} \tag{20}$$

We may refer to equation 18 as *Vlaenderen's law of induction* [1] and equation 19 as *Nikolaev's law of induction* [3]. And equation 20 is obviously *Faraday's law of induction.*

With the definitions in this chapter and the definitions in appendix A and B we are ready to state and prove our theorems. The equation numbering in the theorems is with respect to theorem number.

## 2 Theorems

**Theorem 1.** *The magnetic vector field* $\mathbf{B}_T$ *as a function of* $\mathbf{x}$ *is given by the Biot-Savart Law*

$$\boxed{\mathbf{B}_T(\mathbf{x}) = \frac{\mu_0}{4\pi} \int\limits_{V'} \frac{1}{r^3} \left(\mathbf{J}(\mathbf{x}') \times \mathbf{r}\right) \, \mathrm{d}^3 x'} \tag{1.1}$$

*Proof.*

$$\begin{aligned}
\mathbf{B}_T(\mathbf{x}) &= \nabla \times \mathbf{A}(\mathbf{x}) = \nabla \times \mathbf{A}_t(\mathbf{x}) && (1.2)\\
&= \nabla \times \left( \frac{\mu_0}{4\pi} \int\limits_{V'} \frac{\mathbf{J}(\mathbf{x}')}{r} \, \mathrm{d}^3 x' \right) && (1.3)\\
&= \frac{\mu_0}{4\pi} \int\limits_{V'} \nabla \times \left( \frac{\mathbf{J}(\mathbf{x}')}{r} \right) \, \mathrm{d}^3 x' && (1.4)\\
&= \frac{\mu_0}{4\pi} \int\limits_{V'} \left( \frac{1}{r} \right) \nabla \times \mathbf{J}(\mathbf{x}') + \nabla \left( \frac{1}{r} \right) \times \mathbf{J}(\mathbf{x}') \, \mathrm{d}^3 x' && (1.5)\\
&= -\frac{\mu_0}{4\pi} \int\limits_{V'} \mathbf{J}(\mathbf{x}') \times \nabla \left( \frac{1}{r} \right) \, \mathrm{d}^3 x' && (1.6)\\
&= -\frac{\mu_0}{4\pi} \int\limits_{V'} \mathbf{J}(\mathbf{x}') \times \left( -\frac{\hat{\mathbf{r}}}{r^2} \right) \, \mathrm{d}^3 x' && (1.7)\\
&= \frac{\mu_0}{4\pi} \int\limits_{V'} \frac{\mathbf{J}(\mathbf{x}') \times \hat{\mathbf{r}}}{r^2} \, \mathrm{d}^3 x' && (1.8)\\
&= \frac{\mu_0}{4\pi} \int\limits_{V'} \frac{\mathbf{J}(\mathbf{x}') \times \mathbf{r}}{r^3} \, \mathrm{d}^3 x' && (1.9)\\
\mathbf{B}_T(\mathbf{x}) &= \frac{\mu_0}{4\pi} \int\limits_{V'} \frac{1}{r^3} \left(\mathbf{J}(\mathbf{x}') \times \mathbf{r}\right) \, \mathrm{d}^3 x' && (1.10)
\end{aligned}$$

■

**Theorem 2.** *The magnetic scalar field $B_L$ as a function of $\mathbf{x}$ is given by*

$$\boxed{B_L(\mathbf{x}) = \frac{\mu_0}{4\pi} \int\limits_{V'} \frac{1}{r^3} \big(\mathbf{J}(\mathbf{x}') \cdot \mathbf{r}\big) \, \mathrm{d}^3x'} \tag{2.1}$$

*Proof.*

$$B_L(\mathbf{x}) = -\nabla\cdot\mathbf{A}(\mathbf{x}) = -\nabla\cdot\mathbf{A}_l(\mathbf{x}) \tag{2.2}$$

$$= -\nabla\cdot\left(\frac{\mu_0}{4\pi}\int\limits_{V'}\frac{\mathbf{J}(\mathbf{x}')}{r}\,\mathrm{d}^3x'\right) \tag{2.3}$$

$$= -\frac{\mu_0}{4\pi}\int\limits_{V'}\nabla\cdot\left(\frac{\mathbf{J}(\mathbf{x}')}{r}\right)\,\mathrm{d}^3x' \tag{2.4}$$

$$= -\frac{\mu_0}{4\pi}\int\limits_{V'}\left(\frac{1}{r}\right)\nabla\cdot\mathbf{J}(\mathbf{x}') + \mathbf{J}(\mathbf{x}')\cdot\nabla\left(\frac{1}{r}\right)\,\mathrm{d}^3x' \tag{2.5}$$

$$= -\frac{\mu_0}{4\pi}\int\limits_{V'}\mathbf{J}(\mathbf{x}')\cdot\nabla\left(\frac{1}{r}\right)\,\mathrm{d}^3x' \tag{2.6}$$

$$= -\frac{\mu_0}{4\pi}\int\limits_{V'}\mathbf{J}(\mathbf{x}')\cdot\left(-\frac{\hat{\mathbf{r}}}{r^2}\right)\,\mathrm{d}^3x' \tag{2.7}$$

$$= \frac{\mu_0}{4\pi}\int\limits_{V'}\frac{\mathbf{J}(\mathbf{x}')\cdot\hat{\mathbf{r}}}{r^2}\,\mathrm{d}^3x' \tag{2.8}$$

$$= \frac{\mu_0}{4\pi}\int\limits_{V'}\frac{\mathbf{J}(\mathbf{x}')\cdot\mathbf{r}}{r^3}\,\mathrm{d}^3x' \tag{2.9}$$

$$B_L(\mathbf{x}) = \frac{\mu_0}{4\pi}\int\limits_{V'}\frac{1}{r^3}\big(\mathbf{J}(\mathbf{x}')\cdot\mathbf{r}\big)\,\mathrm{d}^3x' \tag{2.10}$$

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**Theorem 3.** *Given the magnetic scalar field* $B_L$ *and the magnetic scalar field* $B_\Phi$*, then the gradient of the Lorenz condition equals zero*

$$\boxed{\nabla\big(B_L + \epsilon_0\mu_0 B_\Phi\big) = 0} \tag{3.1}$$

*Proof.*

$$\nabla\cdot\mathbf{E}_\Phi = \frac{1}{\epsilon_0}\rho \tag{3.2}$$

$$\nabla\cdot\mathbf{J}_l = -\frac{\partial\rho}{\partial t} \tag{3.3}$$

$$\nabla\cdot\mathbf{J}_l = -\epsilon_0\frac{\partial(\nabla\cdot\mathbf{E}_\Phi)}{\partial t} \tag{3.4}$$

$$\mathbf{J}_l = -\epsilon_0\frac{\partial(\mathbf{E}_\Phi)}{\partial t} \tag{3.5}$$

$$-\Delta\mathbf{A}(\mathbf{x}) = -\frac{\mu_0}{4\pi}\int\limits_{V'}\Delta\frac{\mathbf{J}(\mathbf{x}')}{r}\,\mathrm{d}^3x' \tag{3.6}$$

$$-\Delta\mathbf{A}(\mathbf{x}) = \mu_0\int\limits_{V'}\delta(\mathbf{x}-\mathbf{x}')\mathbf{J}(\mathbf{x}')\,\mathrm{d}^3x' \tag{3.7}$$

$$-\Delta\mathbf{A}(\mathbf{x}) = \mu_0\mathbf{J}(\mathbf{x}) \tag{3.8}$$

$$-\nabla\nabla\cdot\mathbf{A}_l(\mathbf{x}) + \nabla\times\nabla\times\mathbf{A}_t = \mu_0\mathbf{J}(\mathbf{x}) \tag{3.9}$$

$$\nabla B_L(\mathbf{x}) + \nabla\times\mathbf{B}_T = \mu_0\mathbf{J}(\mathbf{x}) \tag{3.10}$$

$$\nabla B_L(\mathbf{x}) = \mu_0\mathbf{J}_l(\mathbf{x}) \tag{3.11}$$

$$\nabla\times\mathbf{B}_T = \mu_0\mathbf{J}_t(\mathbf{x}) \tag{3.12}$$

$$\nabla\times\mathbf{B}_T - \epsilon_0\mu_0\frac{\partial(\mathbf{E}_\Phi)}{\partial t} = \mu_0\mathbf{J}_t + \mu_0\mathbf{J}_l \tag{3.13}$$

$$\nabla B_L + \nabla\times\mathbf{B}_T - \nabla\times\mathbf{B}_T + \epsilon_0\mu_0\frac{\partial(\mathbf{E}_\Phi)}{\partial t} = \mu_0\mathbf{J} - \mu_0\mathbf{J}_t - \mu_0\mathbf{J}_l \tag{3.14}$$

$$\nabla B_L + \epsilon_0\mu_0\frac{\partial\mathbf{E}_\Phi}{\partial t} = 0 \tag{3.15}$$

$$\nabla B_L + \epsilon_0\mu_0\nabla B_\Phi = 0 \tag{3.16}$$

$$\nabla\big(B_L + \epsilon_0\mu_0 B_\Phi\big) = 0 \tag{3.17}$$

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**Theorem 4.** *Given a charge density $\rho$ and a scalar magnetic field $B_\Phi$, then the power density is given by*

$$\boxed{-\rho B_\Phi = \frac{\phi_0}{2}\frac{\partial B_\Phi^2}{\partial t} + \frac{\epsilon_0}{2}\frac{\partial E_\Phi^2}{\partial t} - \nabla\cdot(B_\Phi \mathbf{E}_\Phi)} \tag{4.1}$$

*Proof.*

$$\begin{aligned}
-\rho B_\Phi &= -\left(\epsilon_0 \nabla\cdot\mathbf{E}_\Phi - \phi_0 \frac{\partial B_\Phi}{\partial t}\right) B_\Phi && (4.2)\\
&= -\epsilon_0(\nabla\cdot\mathbf{E}_\Phi)B_\Phi + \phi_0\frac{\partial B_\Phi}{\partial t}B_\Phi && (4.3)\\
&= -\epsilon_0(\nabla\cdot\mathbf{E}_\Phi)B_\Phi + \frac{\phi_0}{2}\frac{\partial B_\Phi^2}{\partial t} && (4.4)\\
&= -\epsilon_0\left(\nabla\cdot(\mathbf{E}_\Phi B_\Phi) - \nabla B_\Phi \cdot \mathbf{E}_\Phi\right) + \frac{\phi_0}{2}\frac{\partial B_\Phi^2}{\partial t} && (4.5)\\
&= \epsilon_0 \nabla B_\Phi \cdot \mathbf{E}_\Phi - \epsilon_0 \nabla\cdot(\mathbf{E}_\Phi B_\Phi) + \frac{\phi_0}{2}\frac{\partial B_\Phi^2}{\partial t} && (4.6)\\
&= \epsilon_0 \frac{\partial \mathbf{E}_\Phi}{\partial t}\cdot\mathbf{E}_\Phi - \epsilon_0 \nabla\cdot(\mathbf{E}_\Phi B_\Phi) + \frac{\phi_0}{2}\frac{\partial B_\Phi^2}{\partial t} && (4.7)\\
&= \frac{\epsilon_0}{2}\frac{\partial E_\Phi^2}{\partial t} - \epsilon_0 \nabla\cdot(\mathbf{E}_\Phi B_\Phi) + \frac{\phi_0}{2}\frac{\partial B_\Phi^2}{\partial t} && (4.8)\\
-\rho B_\Phi &= \frac{\phi_0}{2}\frac{\partial B_\Phi^2}{\partial t} + \frac{\epsilon_0}{2}\frac{\partial E_\Phi^2}{\partial t} - \epsilon_0 \nabla\cdot(\mathbf{E}_\Phi B_\Phi) && (4.9)
\end{aligned}$$

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**Theorem 5.** *Given a charge density* $\rho$ *and a primary electric field* $\mathbf{E}_\Phi$*, then the force density is given by*

$$\boxed{\rho\mathbf{E}_\Phi = \phi_0(\nabla B_\Phi)B_\Phi + \epsilon_0(\nabla\cdot\mathbf{E}_\Phi)\mathbf{E}_\Phi - \phi_0\frac{\partial(B_\Phi\mathbf{E}_\Phi)}{\partial t}} \tag{5.1}$$

*Proof.*

$$\rho\mathbf{E}_\Phi = \left(\epsilon_0\nabla\cdot\mathbf{E}_\Phi - \phi_0\frac{\partial B_\Phi}{\partial t}\right)\mathbf{E}_\Phi \tag{5.2}$$

$$= \epsilon_0(\nabla\cdot\mathbf{E}_\Phi)\mathbf{E}_\Phi - \phi_0\frac{\partial B_\Phi}{\partial t}\mathbf{E}_\Phi \tag{5.3}$$

$$= \epsilon_0(\nabla\cdot\mathbf{E}_\Phi)\mathbf{E}_\Phi - \phi_0\left(\frac{\partial(B_\Phi\mathbf{E}_\Phi)}{\partial t} - \frac{\partial\mathbf{E}_\Phi}{\partial t}B_\Phi\right) \tag{5.4}$$

$$= \epsilon_0(\nabla\cdot\mathbf{E}_\Phi)\mathbf{E}_\Phi + \phi_0\frac{\partial\mathbf{E}_\Phi}{\partial t}B_\Phi - \phi_0\frac{\partial(B_\Phi\mathbf{E}_\Phi)}{\partial t} \tag{5.5}$$

$$= \epsilon_0(\nabla\cdot\mathbf{E}_\Phi)\mathbf{E}_\Phi + \phi_0(\nabla B_\Phi)B_\Phi - \phi_0\frac{\partial(B_\Phi\mathbf{E}_\Phi)}{\partial t} \tag{5.6}$$

$$\rho\mathbf{E}_\Phi = \phi_0(\nabla B_\Phi)B_\Phi + \epsilon_0(\nabla\cdot\mathbf{E}_\Phi)\mathbf{E}_\Phi - \phi_0\frac{\partial(B_\Phi\mathbf{E}_\Phi)}{\partial t} \tag{5.7}$$

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**Theorem 6.** *Given a curl-free current density* $\mathbf{J}_l$ *and a curl-free induced electric field* $\mathbf{E}_L$*, then the power density is given by*

$$\boxed{-\mathbf{E}_L \cdot \mathbf{J}_l = \frac{1}{2\mu_0}\frac{\partial B_L^2}{\partial t} + \frac{\lambda_0}{2}\frac{\partial E_L^2}{\partial t} - \frac{1}{\mu_0}\nabla\cdot(B_L\mathbf{E}_L)} \tag{6.1}$$

*Proof.*

$$\begin{aligned}
-\mathbf{E}_L \cdot \mathbf{J}_l &= -\mathbf{E}_L \cdot \left(\frac{1}{\mu_0}\nabla B_L - \lambda_0 \frac{\partial \mathbf{E}_L}{\partial t}\right) && (6.2)\\
&= -\frac{1}{\mu_0}\left(\mathbf{E}_L \cdot \nabla B_L\right) + \lambda_0 \left(\mathbf{E}_L \cdot \frac{\partial \mathbf{E}_L}{\partial t}\right) && (6.3)\\
&= -\frac{1}{\mu_0}\left(\mathbf{E}_L \cdot \nabla B_L\right) + \frac{\lambda_0}{2}\frac{\partial E_L^2}{\partial t} && (6.4)\\
&= \frac{1}{\mu_0}\Big(B_L(\nabla\cdot\mathbf{E}_L) - \nabla\cdot(B_L\mathbf{E}_L)\Big) + \frac{\lambda_0}{2}\frac{\partial E_L^2}{\partial t} && (6.5)\\
&= \frac{1}{\mu_0}B_L(\nabla\cdot\mathbf{E}_L) - \frac{1}{\mu_0}\nabla\cdot(B_L\mathbf{E}_L) + \frac{\lambda_0}{2}\frac{\partial E_L^2}{\partial t} && (6.6)\\
&= \frac{1}{\mu_0}B_L\left(\frac{\partial B_L}{\partial t}\right) - \frac{1}{\mu_0}\nabla\cdot(B_L\mathbf{E}_L) + \frac{\lambda_0}{2}\frac{\partial E_L^2}{\partial t} && (6.7)\\
&= \frac{1}{2\mu_0}\frac{\partial B_L^2}{\partial t} - \frac{1}{\mu_0}\nabla\cdot(B_L\mathbf{E}_L) + \frac{\lambda_0}{2}\frac{\partial E_L^2}{\partial t} && (6.8)\\
-\mathbf{E}_L \cdot \mathbf{J}_l &= \frac{1}{2\mu_0}\frac{\partial B_L^2}{\partial t} + \frac{\lambda_0}{2}\frac{\partial E_L^2}{\partial t} - \frac{1}{\mu_0}\nabla\cdot(B_L\mathbf{E}_L) && (6.9)
\end{aligned}$$

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**Theorem 7.** *Given a curl-free current density $\mathbf{J}_l$ and a magnetic scalar field $B_L$, then the force density is given by*

$$\boxed{B_L\mathbf{J}_l = \frac{1}{\mu_0}(\nabla B_L)B_L + \lambda_0(\nabla\cdot\mathbf{E}_L)\mathbf{E}_L - \lambda_0\frac{\partial B_L\mathbf{E}_L}{\partial t}} \tag{7.1}$$

*Proof.*

$$\begin{aligned}
B_L\mathbf{J}_l &= B_L\left(\frac{1}{\mu_0}\nabla B_L - \lambda_0\frac{\partial\mathbf{E}_L}{\partial t}\right) && (7.2)\\
&= \frac{1}{\mu_0}B_L(\nabla B_L) - \lambda_0 B_L\frac{\partial\mathbf{E}_L}{\partial t} && (7.3)\\
&= \frac{1}{\mu_0}B_L(\nabla B_L) - \lambda_0\left(\frac{\partial(B_L\mathbf{E}_L)}{\partial t} - \mathbf{E}_L\frac{\partial B_L}{\partial t}\right) && (7.4)\\
&= \frac{1}{\mu_0}B_L(\nabla B_L) + \lambda_0\mathbf{E}_L\frac{\partial B_L}{\partial t} - \lambda_0\frac{\partial(B_L\mathbf{E}_L)}{\partial t} && (7.5)\\
&= \frac{1}{\mu_0}B_L(\nabla B_L) + \lambda_0\mathbf{E}_L(\nabla\cdot\mathbf{E}_L) - \lambda_0\frac{\partial(B_L\mathbf{E}_L)}{\partial t} && (7.6)\\
B_L\mathbf{J}_l &= \frac{1}{\mu_0}(\nabla B_L)B_L + \lambda_0(\nabla\cdot\mathbf{E}_L)\mathbf{E}_L - \lambda_0\frac{\partial(B_L\mathbf{E}_L)}{\partial t} && (7.7)
\end{aligned}$$

■

**Theorem 8.** *Given a divergence-free current density* $\mathbf{J}_t$ *and a divergence-free induced electric field* $\mathbf{E}_T$*, then the power density is given by*

$$\boxed{-\mathbf{E}_T \cdot \mathbf{J}_t = \frac{1}{2\mu_0}\frac{\partial B_T^2}{\partial t} + \frac{\epsilon_0}{2}\frac{\partial E_T^2}{\partial t} - \frac{1}{\mu_0}\nabla\cdot(\mathbf{B}_T \times \mathbf{E}_T)} \tag{8.1}$$

*Proof.*

$$\begin{aligned}
-\mathbf{E}_T \cdot \mathbf{J}_t &= -\mathbf{E}_T \cdot \left(\frac{1}{\mu_0}\nabla\times\mathbf{B}_T - \epsilon_0\frac{\partial \mathbf{E}_T}{\partial t}\right) && (8.2)\\
&= -\frac{1}{\mu_0}\big(\mathbf{E}_T \cdot \nabla\times\mathbf{B}_T\big) + \epsilon_0\left(\mathbf{E}_T \cdot \frac{\partial \mathbf{E}_T}{\partial t}\right) && (8.3)\\
&= -\frac{1}{\mu_0}\big(\mathbf{E}_T \cdot \nabla\times\mathbf{B}_T\big) + \frac{\epsilon_0}{2}\frac{\partial E_T^2}{\partial t} && (8.4)\\
&= -\frac{1}{\mu_0}\Big(\nabla\cdot(\mathbf{B}_T \times \mathbf{E}_T) + \big(\mathbf{B}_T \cdot \nabla\times\mathbf{E}_T\big)\Big) + \frac{\epsilon_0}{2}\frac{\partial E_T^2}{\partial t} && (8.5)\\
&= -\frac{1}{\mu_0}\nabla\cdot(\mathbf{B}_T \times \mathbf{E}_T) - \frac{1}{\mu_0}\big(\mathbf{B}_T \cdot \nabla\times\mathbf{E}_T\big) + \frac{\epsilon_0}{2}\frac{\partial E_T^2}{\partial t} && (8.6)\\
&= -\frac{1}{\mu_0}\nabla\cdot(\mathbf{B}_T \times \mathbf{E}_T) + \frac{1}{\mu_0}\left(\mathbf{B}_T \cdot \frac{\partial \mathbf{B}_T}{\partial t}\right) + \frac{\epsilon_0}{2}\frac{\partial E_T^2}{\partial t} && (8.7)\\
&= -\frac{1}{\mu_0}\nabla\cdot(\mathbf{B}_T \times \mathbf{E}_T) + \frac{1}{2\mu_0}\frac{\partial B_T^2}{\partial t} + \frac{\epsilon_0}{2}\frac{\partial E_T^2}{\partial t} && (8.8)\\
-\mathbf{E}_T \cdot \mathbf{J}_t &= \frac{1}{2\mu_0}\frac{\partial B_T^2}{\partial t} + \frac{\epsilon_0}{2}\frac{\partial E_T^2}{\partial t} - \frac{1}{\mu_0}\nabla\cdot(\mathbf{B}_T \times \mathbf{E}_T) && (8.9)
\end{aligned}$$

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**Theorem 9.** *Given a divergence-free current density* $\mathbf{J}_t$ *and a divergence-free magnetic vector field* $\mathbf{B}_T$*, then the force density is given by*

$$\boxed{\mathbf{B}_T \times \mathbf{J}_t = \frac{1}{\mu_0}\mathbf{B}_T \times \nabla\times\mathbf{B}_T + \epsilon_0\mathbf{E}_T \times \nabla\times\mathbf{E}_T - \epsilon_0\frac{\partial\mathbf{B}_T \times \mathbf{E}_T}{\partial t}} \tag{9.1}$$

*Proof.*

$$\begin{aligned}
\mathbf{B}_T \times \mathbf{J}_t &= \mathbf{B}_T \times \left(\frac{1}{\mu_0}\nabla\times\mathbf{B}_T - \epsilon_0\frac{\partial\mathbf{E}_T}{\partial t}\right) && (9.2)\\
&= \frac{1}{\mu_0}\mathbf{B}_T \times \nabla\times\mathbf{B}_T - \epsilon_0\left(\mathbf{B}_T \times \frac{\partial\mathbf{E}_T}{\partial t}\right) && (9.3)\\
&= \frac{1}{\mu_0}\mathbf{B}_T \times \nabla\times\mathbf{B}_T - \epsilon_0\left(\frac{\partial\mathbf{B}_T \times \mathbf{E}_T}{\partial t} + \mathbf{E}_T \times \frac{\partial\mathbf{B}_T}{\partial t}\right) && (9.4)\\
&= \frac{1}{\mu_0}\mathbf{B}_T \times \nabla\times\mathbf{B}_T - \epsilon_0\frac{\partial\mathbf{B}_T \times \mathbf{E}_T}{\partial t} - \epsilon_0\left(\mathbf{E}_T \times \frac{\partial\mathbf{B}_T}{\partial t}\right) && (9.5)\\
&= \frac{1}{\mu_0}\mathbf{B}_T \times \nabla\times\mathbf{B}_T - \epsilon_0\frac{\partial\mathbf{B}_T \times \mathbf{E}_T}{\partial t} + \epsilon_0\mathbf{E}_T \times \nabla\times\mathbf{E}_T && (9.6)\\
\mathbf{B}_T \times \mathbf{J}_t &= \frac{1}{\mu_0}\mathbf{B}_T \times \nabla\times\mathbf{B}_T + \epsilon_0\mathbf{E}_T \times \nabla\times\mathbf{E}_T - \epsilon_0\frac{\partial\mathbf{B}_T \times \mathbf{E}_T}{\partial t} && (9.7)
\end{aligned}$$

■

**Theorem 10.** *Given the source terms $\rho$ and $\mathbf{J}$ and the superimposed fields $\mathbf{E}$ and $B$, then the power density can be expressed as*

$$\boxed{\begin{aligned}-\mathbf{E}\cdot\mathbf{J}-c^2B\rho=\frac{1}{\mu_0}&\left(\mathbf{E}\cdot\frac{\partial\mathbf{E}^*}{\partial t}+\mathbf{B}_T\cdot\frac{\partial\mathbf{B}_T}{\partial t}+B\frac{\partial B^*}{\partial t}\right)\\&-\frac{1}{\mu_0}\nabla\cdot(\mathbf{B}_T\times\mathbf{E}+B\mathbf{E})\end{aligned}} \tag{10.1}$$

*Proof.*

$$-\mathbf{E}\cdot\mathbf{J}-c^2B\rho=-\mathbf{E}\cdot\frac{1}{\mu_0}\left(\nabla B+\nabla\times\mathbf{B}_T-\frac{\partial\mathbf{E}^*}{\partial t}\right)-c^2B\left(\epsilon_0\nabla\cdot\mathbf{E}-\epsilon_0\frac{\partial B^*}{\partial t}\right) \tag{10.2}$$

$$=-\frac{1}{\mu_0}\left(\mathbf{E}\cdot\nabla B+\mathbf{E}\cdot\nabla\times\mathbf{B}_T-\mathbf{E}\cdot\frac{\partial\mathbf{E}^*}{\partial t}+B(\nabla\cdot\mathbf{E})-B\frac{\partial B^*}{\partial t}\right) \tag{10.3}$$

$$\begin{aligned}=-\frac{1}{\mu_0}\Bigg(&\nabla\cdot(B\mathbf{E})-B(\nabla\cdot\mathbf{E})+\nabla\cdot(\mathbf{B}_T\times\mathbf{E})+\mathbf{B}_T\cdot\nabla\times\mathbf{E}\\&-\mathbf{E}\cdot\frac{\partial\mathbf{E}^*}{\partial t}+B(\nabla\cdot\mathbf{E})-B\frac{\partial B^*}{\partial t}\Bigg)\end{aligned} \tag{10.4}$$

$$=-\frac{1}{\mu_0}\left(\nabla\cdot(B\mathbf{E})+\nabla\cdot(\mathbf{B}_T\times\mathbf{E})-\mathbf{B}_T\cdot\frac{\partial\mathbf{B}_T}{\partial t}-\mathbf{E}\cdot\frac{\partial\mathbf{E}^*}{\partial t}-B\frac{\partial B^*}{\partial t}\right) \tag{10.5}$$

$$\begin{aligned}-\mathbf{E}\cdot\mathbf{J}-c^2B\rho=&\frac{1}{\mu_0}\left(\mathbf{E}\cdot\frac{\partial\mathbf{E}^*}{\partial t}+\mathbf{B}_T\cdot\frac{\partial\mathbf{B}_T}{\partial t}+B\frac{\partial B^*}{\partial t}\right)\\&-\frac{1}{\mu_0}\nabla\cdot(\mathbf{B}_T\times\mathbf{E}+B\mathbf{E})\end{aligned} \tag{10.6}$$

■

**Theorem 11.** *Given the source terms* $\rho, \mathbf{J}, \mathbf{J}_l, \mathbf{J}_t$*, the vector field* $\mathbf{B}_T$*, the superimposed fields* $\mathbf{E}, \mathbf{E}^*, B^*$ *and constants* $\epsilon_0, \lambda_0$*, then the force density can be expressed as*

$$\boxed{\begin{aligned}\rho\mathbf{E} + \mathbf{J}\times\mathbf{B}_T + (\frac{\epsilon_0}{\lambda_0}\mathbf{J}_l + \mathbf{J}_t)B^* &= \epsilon_0(\nabla\cdot\mathbf{E})\mathbf{E} + \frac{1}{\mu_0}(\nabla\times\mathbf{E})\times\mathbf{E}^* \\ &+ \frac{1}{\mu_0}(\nabla B + \nabla\times\mathbf{B}_T)\times\mathbf{B}_T \\ &+ (\epsilon_0\nabla B_\Phi + \frac{\epsilon_0}{\mu_0\lambda_0}\nabla B_L + \frac{1}{\mu_0}\nabla\times\mathbf{B}_T)B^* \\ &- \epsilon_0\frac{\partial \mathbf{E}B^*}{\partial t} - \frac{1}{\mu_0}\frac{\partial \mathbf{E}^*\times\mathbf{B}_T}{\partial t}\end{aligned}} \tag{11.1}$$

*Proof.*

$$\begin{aligned}\mathbf{J} &= \frac{1}{\mu_0}\left(\nabla B + \nabla\times\mathbf{B}_T - \frac{\partial\mathbf{E}^*}{\partial t}\right) && (11.2)\\ &= \frac{1}{\mu_0}\left(\frac{1}{c^2}\nabla B_\Phi + \nabla B_L + \nabla\times\mathbf{B}_T - \frac{1}{c^2}\frac{\partial\mathbf{E}_\Phi}{\partial t} - \frac{1}{b^2}\frac{\partial\mathbf{E}_L}{\partial t} - \frac{1}{c^2}\frac{\partial\mathbf{E}_T}{\partial t}\right) && (11.3)\\ &= \frac{1}{\mu_0}\nabla B_L + \frac{1}{\mu_0}\nabla\times\mathbf{B}_T - \frac{1}{\mu_0 b^2}\frac{\partial\mathbf{E}_L}{\partial t} - \frac{1}{\mu_0 c^2}\frac{\partial\mathbf{E}_T}{\partial t} && (11.4)\end{aligned}$$

$$\begin{aligned}\mathbf{J}_l &= \frac{1}{\mu_0}\nabla B_L - \lambda_0\frac{\partial\mathbf{E}_L}{\partial t} && (11.5)\\ \mathbf{J}_t &= \frac{1}{\mu_0}\nabla\times\mathbf{B}_T - \epsilon_0\frac{\partial\mathbf{E}_T}{\partial t} && (11.6)\\ \mathbf{J} &= \frac{1}{\mu_0}\nabla B + \frac{1}{\mu_0}\nabla\times\mathbf{B}_T - \frac{1}{\mu_0}\frac{\partial\mathbf{E}^*}{\partial t} && (11.7)\\ \rho &= \epsilon_0\nabla\cdot\mathbf{E} - \epsilon_0\frac{\partial B^*}{\partial t} && (11.8)\end{aligned}$$

$$\begin{aligned}\rho\mathbf{E} &= \epsilon_0(\nabla\cdot\mathbf{E})\mathbf{E} - \epsilon_0\frac{\partial B^*}{\partial t}\mathbf{E} && (11.9)\\ &= \epsilon_0(\nabla\cdot\mathbf{E})\mathbf{E} + \epsilon_0\frac{\partial\mathbf{E}}{\partial t}B^* - \epsilon_0\frac{\partial\mathbf{E}B^*}{\partial t} && (11.10)\end{aligned}$$

$$\begin{aligned}\mathbf{J}\times\mathbf{B}_T &= \frac{1}{\mu_0}(\nabla B)\times\mathbf{B}_T + \frac{1}{\mu_0}(\nabla\times\mathbf{B}_T)\times\mathbf{B}_T - \frac{1}{\mu_0}\frac{\partial\mathbf{E}^*}{\partial t}\times\mathbf{B}_T && (11.11)\\ &= \frac{1}{\mu_0}(\nabla B + \nabla\times\mathbf{B}_T)\times\mathbf{B}_T + \frac{1}{\mu_0}\mathbf{E}^*\times\frac{\partial\mathbf{B}_T}{\partial t} - \frac{1}{\mu_0}\frac{\partial\mathbf{E}^*\times\mathbf{B}_T}{\partial t} && (11.12)\\ &= \frac{1}{\mu_0}(\nabla B + \nabla\times\mathbf{B}_T)\times\mathbf{B}_T + \frac{1}{\mu_0}(\nabla\times\mathbf{E})\times\mathbf{E}^* - \frac{1}{\mu_0}\frac{\partial\mathbf{E}^*\times\mathbf{B}_T}{\partial t} && (11.13)\end{aligned}$$

$$\frac{\epsilon_0}{\lambda_0}\mathbf{J}_l B^* = \frac{\epsilon_0}{\mu_0\lambda_0}(\nabla B_L)B^* - \epsilon_0\frac{\partial \mathbf{E}_L}{\partial t}B^* \tag{11.14}$$

$$\mathbf{J}_t B^* = \frac{1}{\mu_0}(\nabla\times\mathbf{B}_T)B^* - \epsilon_0\frac{\partial \mathbf{E}_T}{\partial t}B^* \tag{11.15}$$

$$\begin{aligned}\rho\mathbf{E} + \mathbf{J}\times\mathbf{B}_T + (\frac{\epsilon_0}{\lambda_0}\mathbf{J}_l + \mathbf{J}_t)B^* = {} & \epsilon_0(\nabla\cdot\mathbf{E})\mathbf{E} + \frac{1}{\mu_0}(\nabla\times\mathbf{E})\times\mathbf{E}^* \\ & + \frac{1}{\mu_0}(\nabla B + \nabla\times\mathbf{B}_T)\times\mathbf{B}_T \\ & + \left(\epsilon_0\left[\frac{\partial\mathbf{E}}{\partial t} - \frac{\partial\mathbf{E}_L}{\partial t} - \frac{\partial\mathbf{E}_T}{\partial t}\right] + \frac{\epsilon_0}{\mu_0\lambda_0}\nabla B_L + \frac{1}{\mu_0}\nabla\times\mathbf{B}_T\right)B^* \\ & - \epsilon_0\frac{\partial\mathbf{E}B^*}{\partial t} - \frac{1}{\mu_0}\frac{\partial\mathbf{E}^*\times\mathbf{B}_T}{\partial t}\end{aligned} \tag{11.16}$$

$$\begin{aligned}\rho\mathbf{E} + \mathbf{J}\times\mathbf{B}_T + (\frac{\epsilon_0}{\lambda_0}\mathbf{J}_l + \mathbf{J}_t)B^* = {} & \epsilon_0(\nabla\cdot\mathbf{E})\mathbf{E} + \frac{1}{\mu_0}(\nabla\times\mathbf{E})\times\mathbf{E}^* \\ & + \frac{1}{\mu_0}(\nabla B + \nabla\times\mathbf{B}_T)\times\mathbf{B}_T \\ & + \left(\epsilon_0\nabla B_\Phi + \frac{\epsilon_0}{\mu_0\lambda_0}\nabla B_L + \frac{1}{\mu_0}\nabla\times\mathbf{B}_T\right)B^* \\ & - \epsilon_0\frac{\partial\mathbf{E}B^*}{\partial t} - \frac{1}{\mu_0}\frac{\partial\mathbf{E}^*\times\mathbf{B}_T}{\partial t}\end{aligned} \tag{11.17}$$

■

**Theorem 12.** *Given the source terms $\rho, \mathbf{J}$ and superimposed fields $\mathbf{E}$ and $B$, then the power density is given by*

$$\boxed{-\mathbf{E}\cdot\mathbf{J} - c^2 B\rho = \frac{\epsilon_0}{2}\frac{\partial E^2}{\partial t} + \frac{1}{2\mu_0}\frac{\partial B_T^2}{\partial t} + \frac{B}{\mu_0}\frac{\partial B^*}{\partial t} - \frac{1}{\mu_0}\nabla\cdot(\mathbf{B}_T \times \mathbf{E} + B\mathbf{E})} \tag{12.1}$$

*Proof.*

$$\begin{aligned}
-\mathbf{E}\cdot\mathbf{J} - c^2 B\rho &= -\frac{1}{\mu_0}\mathbf{E}\cdot\left(\nabla B + \nabla\times\mathbf{B}_T - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\right) - \epsilon_0 c^2 B\left(\nabla\cdot\mathbf{E} - \frac{\partial B^*}{\partial t}\right) && (12.2)\\
&= -\frac{1}{\mu_0}\left(\mathbf{E}\cdot(\nabla B) + \mathbf{E}\cdot(\nabla\times\mathbf{B}_T) - \frac{1}{c^2}\mathbf{E}\cdot\frac{\partial \mathbf{E}}{\partial t} + B(\nabla\cdot\mathbf{E}) - B\frac{\partial B^*}{\partial t}\right) && (12.3)\\
&= \frac{\epsilon_0}{2}\frac{\partial E^2}{\partial t} + \frac{B}{\mu_0}\frac{\partial B^*}{\partial t} - \frac{1}{\mu_0}\Big(\mathbf{E}\cdot(\nabla B) + \mathbf{E}\cdot(\nabla\times\mathbf{B}_T) + B(\nabla\cdot\mathbf{E})\Big) && (12.4)\\
&= \frac{\epsilon_0}{2}\frac{\partial E^2}{\partial t} + \frac{B}{\mu_0}\frac{\partial B^*}{\partial t} && (12.5)\\
&\quad - \frac{1}{\mu_0}\Big(\nabla\cdot(B\mathbf{E}) - B(\nabla\cdot\mathbf{E}) + \nabla\cdot(\mathbf{B}_T\times\mathbf{E}) + \mathbf{B}_T\cdot(\nabla\times\mathbf{E}) + B(\nabla\cdot\mathbf{E})\Big)\\
&= \frac{\epsilon_0}{2}\frac{\partial E^2}{\partial t} + \frac{B}{\mu_0}\frac{\partial B^*}{\partial t} - \frac{1}{\mu_0}\Big(\nabla\cdot(B\mathbf{E}) + \nabla\cdot(\mathbf{B}_T\times\mathbf{E}) + \mathbf{B}_T\cdot(\nabla\times\mathbf{E})\Big) && (12.6)\\
&= \frac{\epsilon_0}{2}\frac{\partial E^2}{\partial t} + \frac{1}{2\mu_0}\frac{\partial B_T^2}{\partial t} + \frac{B}{\mu_0}\frac{\partial B^*}{\partial t} - \frac{1}{\mu_0}\Big(\nabla\cdot(B\mathbf{E}) + \nabla\cdot(\mathbf{B}_T\times\mathbf{E})\Big) && (12.7)\\
-\mathbf{E}\cdot\mathbf{J} - c^2 B\rho &= \frac{\epsilon_0}{2}\frac{\partial E^2}{\partial t} + \frac{1}{2\mu_0}\frac{\partial B_T^2}{\partial t} + \frac{B}{\mu_0}\frac{\partial B^*}{\partial t} - \frac{1}{\mu_0}\nabla\cdot(\mathbf{B}_T\times\mathbf{E} + B\mathbf{E}) && (12.8)
\end{aligned}$$

■

**Theorem 13.** *Given the source terms $\rho$ and $\mathbf{J}$, the superimposed fields $\mathbf{E}, B, B^*$ and the magnetic vector field $\mathbf{B}_T$, then the force density is given by*

$$\boxed{\begin{aligned}\rho\mathbf{E} + \mathbf{J}\times\mathbf{B}_T + \mathbf{J}B^* = \epsilon_0\big((\nabla\cdot\mathbf{E})\mathbf{E} + (\nabla\times\mathbf{E})\times\mathbf{E}\big)\\ + \frac{1}{\mu_0}(\nabla B + \nabla\times\mathbf{B}_T)\times\mathbf{B}_T\\ + \frac{1}{\mu_0}(\nabla B + \nabla\times\mathbf{B}_T)B^*\\ - \epsilon_0\frac{\partial(\mathbf{E}B^* + \mathbf{E}\times\mathbf{B}_T)}{\partial t}\end{aligned}} \tag{13.1}$$

*Proof.*

$$\rho\mathbf{E} = \epsilon_0(\nabla\cdot\mathbf{E})\mathbf{E} + \epsilon_0\frac{\partial\mathbf{E}}{\partial t}B^* - \epsilon_0\frac{\partial\mathbf{E}B^*}{\partial t} \tag{13.2}$$

$$\begin{aligned}\mathbf{J}\times\mathbf{B}_T &= \frac{1}{\mu_0}(\nabla B)\times\mathbf{B}_T + \frac{1}{\mu_0}(\nabla\times\mathbf{B}_T)\times\mathbf{B}_T - \frac{1}{\mu_0 c^2}\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B}_T &(13.3)\\ &= \frac{1}{\mu_0}(\nabla B + \nabla\times\mathbf{B}_T)\times\mathbf{B}_T + \frac{1}{\mu_0 c^2}\mathbf{E}\times\frac{\partial\mathbf{B}_T}{\partial t} - \frac{1}{\mu_0 c^2}\frac{\partial\mathbf{E}\times\mathbf{B}_T}{\partial t} &(13.4)\\ &= \frac{1}{\mu_0}(\nabla B + \nabla\times\mathbf{B}_T)\times\mathbf{B}_T + \epsilon_0(\nabla\times\mathbf{E})\times\mathbf{E} - \epsilon_0\frac{\partial\mathbf{E}\times\mathbf{B}_T}{\partial t} &(13.5)\end{aligned}$$

$$\begin{aligned}\mathbf{J}B^* &= \frac{1}{\mu_0}\left(\nabla B + \nabla\times\mathbf{B}_T - \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}\right)B^* &(13.6)\\ &= \frac{1}{\mu_0}(\nabla B + \nabla\times\mathbf{B}_T)B^* - \frac{1}{\mu_0 c^2}\frac{\partial\mathbf{E}}{\partial t}B^* &(13.7)\\ &= \frac{1}{\mu_0}(\nabla B + \nabla\times\mathbf{B}_T)B^* + \frac{1}{\mu_0 c^2}\frac{\partial B^*}{\partial t}\mathbf{E} - \frac{1}{\mu_0 c^2}\frac{\partial\mathbf{E}B^*}{\partial t} &(13.8)\\ &= \frac{1}{\mu_0}(\nabla B + \nabla\times\mathbf{B}_T)B^* + \epsilon_0\frac{\partial B^*}{\partial t}\mathbf{E} - \epsilon_0\frac{\partial\mathbf{E}B^*}{\partial t} &(13.9)\end{aligned}$$

$$\begin{aligned}\rho\mathbf{E} + \mathbf{J}\times\mathbf{B}_T + \mathbf{J}B^* &= \epsilon_0(\nabla\cdot\mathbf{E})\mathbf{E} + \epsilon_0\frac{\partial\mathbf{E}}{\partial t}B^* - \epsilon_0\frac{\partial\mathbf{E}B^*}{\partial t} &(13.10)\\ &+ \frac{1}{\mu_0}(\nabla B + \nabla\times\mathbf{B}_T)\times\mathbf{B}_T + \epsilon_0(\nabla\times\mathbf{E})\times\mathbf{E} - \epsilon_0\frac{\partial\mathbf{E}\times\mathbf{B}_T}{\partial t}\\ &+ \frac{1}{\mu_0}(\nabla B + \nabla\times\mathbf{B}_T)B^* + \epsilon_0\frac{\partial B^*}{\partial t}\mathbf{E} - \epsilon_0\frac{\partial\mathbf{E}B^*}{\partial t}\\ &= \epsilon_0\big((\nabla\cdot\mathbf{E})\mathbf{E} + (\nabla\times\mathbf{E})\times\mathbf{E}\big) &(13.11)\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\mu_0}(\nabla B+\nabla\times\mathbf{B}_T)\times\mathbf{B}_T\\
&+\frac{1}{\mu_0}(\nabla B+\nabla\times\mathbf{B}_T)B^*\\
&+\epsilon_0\frac{\partial\mathbf{E}}{\partial t}B^*+\epsilon_0\frac{\partial B^*}{\partial t}\mathbf{E}-2\epsilon_0\frac{\partial\mathbf{E}B^*}{\partial t}-\epsilon_0\frac{\partial\mathbf{E}\times\mathbf{B}_T}{\partial t}\\
&-\epsilon_0\frac{\partial\mathbf{E}B^*}{\partial t}-\epsilon_0\frac{\partial\mathbf{E}\times\mathbf{B}_T}{\partial t}
\end{aligned}
$$

$$
\begin{aligned}
\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}_T+\mathbf{J}B^* &= \epsilon_0\big((\nabla\cdot\mathbf{E})\mathbf{E}+(\nabla\times\mathbf{E})\times\mathbf{E}\big)\\
&+\frac{1}{\mu_0}(\nabla B+\nabla\times\mathbf{B}_T)\times\mathbf{B}_T\\
&+\frac{1}{\mu_0}(\nabla B+\nabla\times\mathbf{B}_T)B^*\\
&-\epsilon_0\frac{\partial(\mathbf{E}B^*+\mathbf{E}\times\mathbf{B}_T)}{\partial t}
\end{aligned}
\tag{13.12}
$$

■

**Theorem 14.** *Given the source term* $\mathbf{J}$ *and the primary electric field* $\mathbf{E}_\Phi$*, then the magnetostatic power density is given by*

$$\boxed{-\mathbf{E}_\Phi \cdot \mathbf{J} = \frac{\epsilon_0}{2}\frac{\partial E_\Phi^2}{\partial t} + \frac{1}{\mu_0}\nabla\cdot(\mathbf{E}_\Phi \times \mathbf{B}_T)} \tag{14.1}$$

*Proof.*

$$\mathbf{E}^* = \frac{1}{c^2}\mathbf{E} = \frac{1}{c^2}\mathbf{E}_\Phi \tag{14.2}$$

$$B = 0 \tag{14.3}$$

$$-\mathbf{E}_\Phi \cdot \mathbf{J} = -\mathbf{E}_\Phi \cdot \frac{1}{\mu_0}\left(\nabla\times\mathbf{B}_T - \frac{1}{c^2}\frac{\partial \mathbf{E}_\Phi}{\partial t}\right) \tag{14.4}$$

$$= -\frac{1}{\mu_0}\mathbf{E}_\Phi \cdot \nabla\times\mathbf{B}_T + \frac{1}{\mu_0 c^2}\mathbf{E}_\Phi \cdot \frac{\partial \mathbf{E}_\Phi}{\partial t} \tag{14.5}$$

$$= \frac{1}{\mu_0}\mathbf{B}_T \cdot \nabla\times\mathbf{E}_\Phi + \frac{1}{\mu_0}\nabla\cdot(\mathbf{E}_\Phi \times \mathbf{B}_T) + \frac{\epsilon_0}{2}\frac{\partial E_\Phi^2}{\partial t} \tag{14.6}$$

$$-\mathbf{E}_\Phi \cdot \mathbf{J} = \frac{\epsilon_0}{2}\frac{\partial E_\Phi^2}{\partial t} + \frac{1}{\mu_0}\nabla\cdot(\mathbf{E}_\Phi \times \mathbf{B}_T) \tag{14.7}$$

■

**Theorem 15.** *Given the source terms $\rho$, $\mathbf{J}$, the fields $\mathbf{E}_\Phi$,$\mathbf{B}_T$,$B_L$ and velocities $a$ and $c$, then the magnetostatic force density is given by*

$$\boxed{\begin{aligned}\rho\mathbf{E}_\Phi + \mathbf{J}\times\mathbf{B}_T + (1-\frac{c^2}{a^2})\mathbf{J}B_L = {} & \epsilon_0(\nabla\cdot\mathbf{E}_\Phi)\mathbf{E}_\Phi \\ & + \frac{1}{\mu_0}\Big((\nabla\times\mathbf{B}_T)\times\mathbf{B}_T + (1-\frac{c^2}{a^2})(\nabla\times\mathbf{B}_T)B_L\Big) \\ & - \epsilon_0\Big((1-\frac{c^2}{a^2})\frac{\partial\mathbf{E}_\Phi}{\partial t}B_L + \frac{\partial\mathbf{E}_\Phi}{\partial t}\times\mathbf{B}_T\Big)\end{aligned}} \tag{15.1}$$

*Proof.*

$$\begin{aligned}\mathbf{E}^* &= \frac{1}{c^2}\mathbf{E} = \frac{1}{c^2}\mathbf{E}_\Phi && (15.2)\\ B &= \frac{1}{c^2}B_\Phi + B_L = 0 && (15.3)\\ B_\Phi &= -c^2 B_L && (15.4)\\ B^* &= \frac{1}{a^2}B_\Phi + B_L && (15.5)\\ &= -\frac{c^2}{a^2}B_L + B_L && (15.6)\\ B^* &= (1-\frac{c^2}{a^2})B_L && (15.7)\end{aligned}$$

$$\begin{aligned}\rho\mathbf{E}_\Phi &= \epsilon_0(\nabla\cdot\mathbf{E}_\Phi)\mathbf{E}_\Phi && (15.8)\\ \mathbf{J}\times\mathbf{B}_T &= \frac{1}{\mu_0}\Big(\nabla\times\mathbf{B}_T - \frac{1}{c^2}\frac{\partial\mathbf{E}_\Phi}{\partial t}\Big)\times\mathbf{B}_T && (15.9)\\ &= \frac{1}{\mu_0}(\nabla\times\mathbf{B}_T)\times\mathbf{B}_T - \epsilon_0\frac{\partial\mathbf{E}_\Phi}{\partial t}\times\mathbf{B}_T && (15.10)\\ (1-\frac{c^2}{a^2})\mathbf{J}B_L &= (1-\frac{c^2}{a^2})\frac{1}{\mu_0}\Big(\nabla\times\mathbf{B}_T - \frac{1}{c^2}\frac{\partial\mathbf{E}_\Phi}{\partial t}\Big)B_L && (15.11)\\ &= \frac{1}{\mu_0}(1-\frac{c^2}{a^2})(\nabla\times\mathbf{B}_T)B_L - \epsilon_0(1-\frac{c^2}{a^2})\frac{\partial\mathbf{E}_\Phi}{\partial t}B_L && (15.12)\end{aligned}$$

$$\rho\mathbf{E}_\Phi + \mathbf{J}\times\mathbf{B}_T + (1-\frac{c^2}{a^2})\mathbf{J}B_L = \epsilon_0(\nabla\cdot\mathbf{E}_\Phi)\mathbf{E}_\Phi \tag{15.13}$$

$$
\begin{aligned}
&+\frac{1}{\mu_0}\left((\nabla\times\mathbf{B}_T)\times\mathbf{B}_T+(1-\frac{c^2}{a^2})(\nabla\times\mathbf{B}_T)B_L\right)\\
&-\epsilon_0\left((1-\frac{c^2}{a^2})\frac{\partial\mathbf{E}_\Phi}{\partial t}B_L+\frac{\partial\mathbf{E}_\Phi}{\partial t}\times\mathbf{B}_T\right)
\end{aligned}
$$

■

**Theorem 16.** *The magnetic scalar field $B_\Phi$ as a function of position $\mathbf{x}$ and time $t$ is given by*

$$\boxed{B_\Phi(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\int\limits_{V'} \frac{-\dot{\rho}(\mathbf{x}',t_a)}{r}\,\mathrm{d}^3x'} \tag{16.1}$$

*Proof.*

$$\begin{aligned}
B_\Phi(\mathbf{x},t) &= -\frac{\partial\Phi(\mathbf{x},t)}{\partial t} && (16.2)\\
&= -\frac{\partial}{\partial t}\left(\frac{1}{4\pi\epsilon_0}\int\limits_{V'} \frac{\rho(\mathbf{x}',t_a)}{r}\,\mathrm{d}^3x'\right) && (16.3)\\
&= -\frac{1}{4\pi\epsilon_0}\int\limits_{V'} \frac{1}{r}\left(\frac{\partial\rho}{\partial t_a}\frac{\partial t_a}{\partial t}\right)\mathrm{d}^3x' && (16.4)\\
&= -\frac{1}{4\pi\epsilon_0}\int\limits_{V'} \frac{1}{r}\dot{\rho}\,\mathrm{d}^3x' && (16.5)\\
&= \frac{1}{4\pi\epsilon_0}\int\limits_{V'} \frac{-\dot{\rho}(\mathbf{x}',t_a)}{r}\,\mathrm{d}^3x' && (16.6)
\end{aligned}$$

■

**Theorem 17.** *The magnetic scalar field $B_L$ as a function of position $\mathbf{x}$ and time $t$ is given by*

$$\boxed{B_L(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int\limits_{V'}\left[\frac{\mathbf{J}_l(\mathbf{x}',t_c)\cdot\mathbf{r}}{r^3} + \frac{\dot{\mathbf{J}}_l(\mathbf{x}',t_c)\cdot\mathbf{r}}{cr^2}\right]\mathrm{d}^3x'} \tag{17.1}$$

*Proof.*

$$\begin{aligned}
\nabla\cdot\mathbf{J}_l(\mathbf{x}',t_c) &= \nabla\cdot\mathbf{J}_l(t_c) && (17.2)\\
&= \left(\frac{\partial J_x(t_c)}{\partial x} + \frac{\partial J_y(t_c)}{\partial y} + \frac{\partial J_z(t_c)}{\partial z}\right) && (17.3)\\
&= \left(\frac{\partial J_x}{\partial t_c}\frac{\partial t_c}{\partial x} + \frac{\partial J_y}{\partial t_c}\frac{\partial t_c}{\partial y} + \frac{\partial J_z}{\partial t_c}\frac{\partial t_c}{\partial z}\right) && (17.4)\\
&= \left(\dot{J}_x\frac{\partial t_c}{\partial x} + \dot{J}_y\frac{\partial t_c}{\partial y} + \dot{J}_z\frac{\partial t_c}{\partial z}\right) && (17.5)\\
&= -\frac{1}{c}\left(\dot{J}_x\frac{\partial r}{\partial x} + \dot{J}_y\frac{\partial r}{\partial y} + \dot{J}_z\frac{\partial r}{\partial z}\right) && (17.6)\\
&= -\frac{1}{c}\dot{\mathbf{J}}_l\cdot\nabla r = -\frac{1}{c}\dot{\mathbf{J}}_l\cdot\hat{\mathbf{r}} = -\frac{1}{c}\frac{\dot{\mathbf{J}}_l\cdot\mathbf{r}}{r} && (17.7)\\
\nabla\cdot\mathbf{J}_l(\mathbf{x}',t_c) &= -\frac{\dot{\mathbf{J}}_l(\mathbf{x}',t_c)\cdot\mathbf{r}}{cr} && (17.8)
\end{aligned}$$

$$\begin{aligned}
B_L(\mathbf{x},t) &= -\nabla\cdot\mathbf{A}_l(\mathbf{x},t) && (17.9)\\
&= -\nabla\cdot\left(\frac{\mu_0}{4\pi}\int\limits_{V'}\frac{\mathbf{J}_l(\mathbf{x}',t_c)}{r}\,\mathrm{d}^3x'\right) && (17.10)\\
&= -\frac{\mu_0}{4\pi}\int\limits_{V'}\nabla\cdot\left(\frac{\mathbf{J}_l(\mathbf{x}',t_c)}{r}\right)\mathrm{d}^3x' && (17.11)\\
&= -\frac{\mu_0}{4\pi}\int\limits_{V'}\left[\frac{1}{r}(\nabla\cdot\mathbf{J}_l) + \mathbf{J}_l\cdot\nabla\left(\frac{1}{r}\right)\right]\mathrm{d}^3x' && (17.12)\\
&= -\frac{\mu_0}{4\pi}\int\limits_{V'}\left[\mathbf{J}_l\cdot\nabla\left(\frac{1}{r}\right) + \frac{1}{r}(\nabla\cdot\mathbf{J}_l)\right]\mathrm{d}^3x' && (17.13)\\
&= -\frac{\mu_0}{4\pi}\int\limits_{V'}\left[-\mathbf{J}_l(\mathbf{x}',t_c)\cdot\left(\frac{\mathbf{r}}{r^3}\right) + \frac{1}{r}(\nabla\cdot\mathbf{J}_l)\right]\mathrm{d}^3x' && (17.14)\\
&= -\frac{\mu_0}{4\pi}\int\limits_{V'}\left[-\mathbf{J}_l(\mathbf{x}',t_c)\cdot\left(\frac{\mathbf{r}}{r^3}\right) - \frac{1}{r}\frac{\dot{\mathbf{J}}_l(\mathbf{x}',t_c)\cdot\mathbf{r}}{cr}\right]\mathrm{d}^3x' && (17.15)
\end{aligned}$$

$$B_L(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int\limits_{V'}\left[\frac{\mathbf{J}_l(\mathbf{x}',t_c)\cdot\mathbf{r}}{r^3} + \frac{\dot{\mathbf{J}}_l(\mathbf{x}',t_c)\cdot\mathbf{r}}{cr^2}\right]\mathrm{d}^3x' \tag{17.16}$$

■

**Theorem 18.** *The magnetic vector field* $\mathbf{B}_T$ *as a function of position* $\mathbf{x}$ *and time* $t$ *is given by*

$$\boxed{\mathbf{B}_T(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int\limits_{V'} \left[\frac{\mathbf{J}_t(\mathbf{x}',t_c)\times\mathbf{r}}{r^3} + \frac{\dot{\mathbf{J}}_t(\mathbf{x}',t_c)\times\mathbf{r}}{cr^2}\right]\mathrm{d}^3x'} \tag{18.1}$$

*Proof.*

$$\nabla\times\mathbf{J}_t(\mathbf{x}',t_c) = \nabla\times\mathbf{J}_t(t_c) \tag{18.2}$$

$$(\nabla\times\mathbf{J}_t)_x = \left(\frac{\partial J_z(t_c)}{\partial y} - \frac{\partial J_y(t_c)}{\partial z}\right) \tag{18.3}$$

$$= \left(\frac{\partial J_z}{\partial t_c}\frac{\partial t_c}{\partial y} - \frac{\partial J_y}{\partial t_c}\frac{\partial t_c}{\partial z}\right) \tag{18.4}$$

$$= \left(\dot{J}_z\frac{\partial t_c}{\partial y} - \dot{J}_y\frac{\partial t_c}{\partial z}\right) \tag{18.5}$$

$$= -\frac{1}{c}\left(\dot{J}_z\frac{\partial r}{\partial y} - \dot{J}_y\frac{\partial r}{\partial z}\right) \tag{18.6}$$

$$= -\frac{1}{c}\left(\dot{J}_z(\nabla r)_y - \dot{J}_y(\nabla r)_z\right) \tag{18.7}$$

$$= -\frac{1}{c}(\nabla r\times\dot{\mathbf{J}}_t)_x \tag{18.8}$$

$$(\nabla\times\mathbf{J}_t)_x = \frac{1}{c}(\dot{\mathbf{J}}_t\times\nabla r)_x = \frac{1}{c}(\dot{\mathbf{J}}_t\times\hat{\mathbf{r}})_x \tag{18.9}$$

$$\nabla\times\mathbf{J}_t = \frac{1}{c}(\dot{\mathbf{J}}_t\times\hat{\mathbf{r}}) = \frac{1}{c}\frac{\dot{\mathbf{J}}_t\times\mathbf{r}}{r} \tag{18.10}$$

$$\nabla\times\mathbf{J}_t(\mathbf{x}',t_c) = \frac{\dot{\mathbf{J}}_t\times\mathbf{r}}{cr} \tag{18.11}$$

$$\mathbf{B}_T(\mathbf{x},t) = \nabla\times\mathbf{A}_t(\mathbf{x},t) \tag{18.12}$$

$$= \nabla\times\left(\frac{\mu_0}{4\pi}\int\limits_{V'}\frac{\mathbf{J}_t(\mathbf{x}',t_c)}{r}\,\mathrm{d}^3x'\right) \tag{18.13}$$

$$= \frac{\mu_0}{4\pi}\int\limits_{V'}\nabla\times\left(\frac{\mathbf{J}_t(\mathbf{x}',t_c)}{r}\right)\mathrm{d}^3x' \tag{18.14}$$

$$= \frac{\mu_0}{4\pi}\int\limits_{V'}\left[\frac{1}{r}(\nabla\times\mathbf{J}_t) + \nabla\left(\frac{1}{r}\right)\times\mathbf{J}_t\right]\mathrm{d}^3x' \tag{18.15}$$

$$= \frac{\mu_0}{4\pi} \int\limits_{V'} \left[ \nabla\left(\frac{1}{r}\right) \times \mathbf{J}_t + \frac{1}{r}(\nabla\times\mathbf{J}_t) \right] \mathrm{d}^3x' \tag{18.16}$$

$$= \frac{\mu_0}{4\pi} \int\limits_{V'} \left[ -\left(\frac{\mathbf{r}}{r^3}\right) \times \mathbf{J}_t(\mathbf{x}', t_c) + \frac{1}{r}(\nabla\times\mathbf{J}_t) \right] \mathrm{d}^3x' \tag{18.17}$$

$$= \frac{\mu_0}{4\pi} \int\limits_{V'} \left[ \mathbf{J}_t(\mathbf{x}', t_c) \times \left(\frac{\mathbf{r}}{r^3}\right) + \frac{1}{r}(\nabla\times\mathbf{J}_t) \right] \mathrm{d}^3x' \tag{18.18}$$

$$= \frac{\mu_0}{4\pi} \int\limits_{V'} \left[ \mathbf{J}_t(\mathbf{x}', t_c) \times \left(\frac{\mathbf{r}}{r^3}\right) + \frac{1}{r}\frac{\dot{\mathbf{J}}_t \times \mathbf{r}}{cr} \right] \mathrm{d}^3x' \tag{18.19}$$

$$\mathbf{B}_T(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \int\limits_{V'} \left[ \frac{\mathbf{J}_t(\mathbf{x}', t_c) \times \mathbf{r}}{r^3} + \frac{\dot{\mathbf{J}}_t(\mathbf{x}', t_c) \times \mathbf{r}}{cr^2} \right] \mathrm{d}^3x' \tag{18.20}$$

■

**Theorem 19.** *The the magnetic scalar field* $\mathbf{E}$ *as a function of position vector* $\mathbf{x}$ *and time* $t$ *is given by*

$$\boxed{\mathbf{E}(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\int\limits_{V'}\left[\frac{\rho(\mathbf{x}',t_a)\mathbf{r}}{r^3} + \frac{\dot{\rho}(\mathbf{x}',t_a)\mathbf{r}}{ar^2} - \frac{\dot{\mathbf{J}}_l(\mathbf{x}',t_c)}{c^2 r} - \frac{\dot{\mathbf{J}}_t(\mathbf{x}',t_c)}{c^2 r}\right]\mathrm{d}^3x'} \tag{19.1}$$

*Proof.*

$$\begin{aligned}
\mathbf{E}(\mathbf{x},t) &= -\nabla\Phi - \frac{\partial \mathbf{A}_l}{\partial t} - \frac{\partial \mathbf{A}_t}{\partial t} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} && (19.2)\\
&= -\nabla\left(\frac{1}{4\pi\epsilon_0}\int\limits_{V'}\frac{\rho(\mathbf{x}',t_a)}{r}\,\mathrm{d}^3x'\right) - \frac{\partial}{\partial t}\left(\frac{\mu_0}{4\pi}\int\limits_{V'}\frac{\mathbf{J}(\mathbf{x}',t_c)}{r}\,\mathrm{d}^3x'\right) && (19.3)\\
&= -\frac{1}{4\pi\epsilon_0}\int\limits_{V'}\nabla\left(\frac{\rho}{r}\right)\mathrm{d}^3x' - \frac{\mu_0}{4\pi}\int\limits_{V'}\frac{\partial}{\partial t}\left(\frac{\mathbf{J}}{r}\right)\mathrm{d}^3x' && (19.4)\\
&= -\frac{1}{4\pi\epsilon_0}\int\limits_{V'}\left[\frac{1}{r}\nabla\rho + \rho\nabla\left(\frac{1}{r}\right)\right]\mathrm{d}^3x' - \frac{\mu_0}{4\pi}\int\limits_{V'}\frac{1}{r}\left(\frac{\partial \mathbf{J}}{\partial t_c}\frac{\partial t_c}{\partial t}\right)\mathrm{d}^3x' && (19.5)\\
&= -\frac{1}{4\pi\epsilon_0}\int\limits_{V'}\left[\frac{1}{r}\nabla\rho + \rho\nabla\left(\frac{1}{r}\right)\right]\mathrm{d}^3x' - \frac{1}{c^2}\frac{1}{4\pi\epsilon_0}\int\limits_{V'}\frac{1}{r}\dot{\mathbf{J}}\,\mathrm{d}^3x' && (19.6)\\
&= -\frac{1}{4\pi\epsilon_0}\int\limits_{V'}\left[\frac{1}{r}\nabla\rho + \rho\nabla\left(\frac{1}{r}\right) + \frac{\dot{\mathbf{J}}}{c^2 r}\right]\mathrm{d}^3x' && (19.7)\\
&= -\frac{1}{4\pi\epsilon_0}\int\limits_{V'}\left[\frac{1}{r}\nabla\rho - \rho\frac{\mathbf{r}}{r^3} + \frac{\dot{\mathbf{J}}}{c^2 r}\right]\mathrm{d}^3x' && (19.8)
\end{aligned}$$

$$\begin{aligned}
\nabla\rho(\mathbf{x}',t_a) &= \nabla\rho(t_a) && (19.9)\\
&= \frac{\partial\rho}{\partial t_a}\nabla t_a = -\frac{1}{a}\frac{\partial\rho}{\partial t_a}\nabla r && (19.10)\\
&= -\frac{1}{a}\frac{\partial\rho}{\partial t}\hat{\mathbf{r}} = -\frac{1}{a}\dot{\rho}\frac{\mathbf{r}}{r} && (19.11)\\
&= -\frac{\dot{\rho}(\mathbf{x}',t_a)\mathbf{r}}{ar} && (19.12)
\end{aligned}$$

$$\begin{aligned}
\mathbf{E}(\mathbf{x},t) &= -\frac{1}{4\pi\epsilon_0}\int\limits_{V'}\left[-\frac{1}{r}\frac{\dot{\rho}(\mathbf{x}',t_a)\mathbf{r}}{ar} - \rho\frac{\mathbf{r}}{r^3} + \frac{\dot{\mathbf{J}}}{c^2 r}\right]\mathrm{d}^3x' && (19.13)\\
&= \frac{1}{4\pi\epsilon_0}\int\limits_{V'}\left[\frac{1}{r}\frac{\dot{\rho}(\mathbf{x}',t_a)\mathbf{r}}{ar} + \rho\frac{\mathbf{r}}{r^3} - \frac{\dot{\mathbf{J}}_l}{c^2 r} - \frac{\dot{\mathbf{J}}_t}{c^2 r}\right]\mathrm{d}^3x' && (19.14)
\end{aligned}$$

$$= \frac{1}{4\pi\epsilon_0} \int\limits_{V'} \left[ \frac{\dot{\rho}(\mathbf{x}', t_a)\mathbf{r}}{ar^2} + \frac{\rho(\mathbf{x}', t_a)\mathbf{r}}{r^3} - \frac{\dot{\mathbf{J}}_l(\mathbf{x}', t_c)}{c^2 r} - \frac{\dot{\mathbf{J}}_t(\mathbf{x}', t_c)}{c^2 r} \right] \mathrm{d}^3x' \qquad (19.15)$$

$$\mathbf{E}(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0} \int\limits_{V'} \left[ \frac{\rho(\mathbf{x}', t_a)\mathbf{r}}{r^3} + \frac{\dot{\rho}(\mathbf{x}', t_a)\mathbf{r}}{ar^2} - \frac{\dot{\mathbf{J}}_l(\mathbf{x}', t_c)}{c^2 r} - \frac{\dot{\mathbf{J}}_t(\mathbf{x}', t_c)}{c^2 r} \right] \mathrm{d}^3x' \qquad (19.16)$$

■

# A Definitions

## A.1 Salars and vectors

Vectors will be bold face. Vectors may have sub-indexes 'l' and 't' for longitudinal and transverse or curl-free and divergence-free according to the fundamental theorem of vector algebra, see **G**[1.3], **G**[1.4] and **G**[1.5]. Fields may have sub-indexes 'L' and 'T' referring to 'LEM' and 'TEM' waves, see [1] section 3 General CED.

| | | |
|---|---|---|
| $\mathbf{v} = (x, y, z)$ | Vectors are bold typed | (A.1) |
| $s\mathbf{v} = (sx, sy, sz)$ | Scalars are regular typed | (A.2) |
| $\mathbf{v} = \mathbf{v}_t + \mathbf{v}_l$ | Vectors may be decomposed into divergence free and curl free parts | (A.3) |
| $\nabla \cdot \mathbf{v}_t = 0$ | Divergence free vector | (A.4) |
| $\nabla \times \mathbf{v}_l = 0$ | Curl free vector | (A.5) |

## A.2 Coordinates, position and displacement

| | | |
|---|---|---|
| $\mathbf{x} = (x, y, z)$ | Position vector in space coordinates | (A.6) |
| $\mathbf{x}' = (x', y', z')$ | Source position vector in space coordinates | (A.7) |
| $\mathbf{r} = \mathbf{x} - \mathbf{x}'$ | Displacement vector | (A.8) |
| $r = \|\mathbf{x} - \mathbf{x}'\|$ | Magnitude of displacement vector | (A.9) |
| $\hat{\mathbf{r}} = \dfrac{\mathbf{x} - \mathbf{x}'}{\|\mathbf{x} - \mathbf{x}'\|}$ | Unit displacement vector | (A.10) |

## A.3 Differentiation

| | | |
|---|---|---|
| $\dfrac{\partial}{\partial t}$ | Partial differential with respect to time | (A.11) |
| $\nabla = \left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}\right)$ | Del operator (in Cartesian coordinates) | (A.12) |
| $\Delta = \nabla \cdot \nabla = \left(\dfrac{\partial^2}{\partial x^2}, \dfrac{\partial^2}{\partial y^2}, \dfrac{\partial^2}{\partial z^2}\right)$ | Laplacian | (A.13) |

# B Vector calculus identities

$$\nabla \cdot (s\mathbf{v}) = s(\nabla \cdot \mathbf{v}) + \mathbf{v} \cdot \nabla s \tag{B.1}$$

$$\nabla\cdot(\frac{\mathbf{v}}{s}) = \frac{1}{s}(\nabla\cdot\mathbf{v}) + \mathbf{v}\cdot\nabla(\frac{1}{s}) \tag{B.2}$$

$$\mathbf{v}\cdot\mathbf{w} = \mathbf{w}\cdot\mathbf{v} \tag{B.3}$$

$$\nabla\times(s\mathbf{v}) = s(\nabla\times\mathbf{v}) + \nabla s\times\mathbf{v} \tag{B.4}$$

$$\nabla\times(\frac{\mathbf{v}}{s}) = \frac{1}{s}(\nabla\times\mathbf{v}) + \nabla(\frac{1}{s})\times\mathbf{v} \tag{B.5}$$

$$\mathbf{v}\times\mathbf{w} = -\mathbf{w}\times\mathbf{v} \tag{B.6}$$

$$\nabla\cdot(\mathbf{v}_1\times\mathbf{v}_2) = \mathbf{v}_2\cdot\nabla\times\mathbf{v}_1 - \mathbf{v}_1\cdot\nabla\times\mathbf{v}_2 \tag{B.7}$$

$$\frac{\partial(\mathbf{v}_1\times\mathbf{v}_2)}{\partial t} = \frac{\partial\mathbf{v}_1}{\partial t}\times\mathbf{v}_2 + \mathbf{v}_1\times\frac{\partial\mathbf{v}_2}{\partial t} \tag{B.8}$$

$$\nabla\left(\frac{1}{r}\right) = -\frac{\mathbf{r}}{r^3} = -\frac{\hat{\mathbf{r}}}{r^2} \tag{B.9}$$

$$\frac{\partial t_a}{\partial t} = \frac{\partial}{\partial t}(t - \frac{r}{a}) = 1 \tag{B.10}$$

$$\partial t_a = \partial t \tag{B.11}$$

$$\frac{\partial}{\partial t_a} = \frac{\partial}{\partial t} \tag{B.12}$$

$$\Delta s = \nabla\cdot\nabla s \tag{B.13}$$

$$\Delta\mathbf{v} = \nabla\nabla\cdot\mathbf{v} - \nabla\times\nabla\times\mathbf{v} \tag{B.14}$$

$$0 = \nabla\cdot\mathbf{v}(\mathbf{x}') = \nabla\times\mathbf{v}(\mathbf{x}') \tag{B.15}$$

$$0 = \nabla'\cdot\mathbf{v}(\mathbf{x}) = \nabla'\times\mathbf{v}(\mathbf{x}) \tag{B.16}$$

# C Vector calculus proofs

## C.1 The gradient of $r$

*Given a displacement vector* $\mathbf{r}$ *with magnitude* $r = |\mathbf{x}-\mathbf{x}'|$ *and unit vector* $\hat{\mathbf{r}}$*, then*

$$\boxed{\nabla r = \hat{\mathbf{r}}} \tag{C.1}$$

*Proof.*

$$(\nabla r)_x = \left(\frac{\partial}{\partial x}\right)r = \left(\frac{\partial}{\partial x}\right)|\mathbf{x}-\mathbf{x}'| \tag{C.2}$$

$$= \left(\frac{\partial}{\partial x}\right)\sqrt{\left((x-x')^2 + (y-y')^2 + (z-z')^2\right)} \tag{C.3}$$

$$= \left(\frac{\partial}{\partial x}\right)\left((x-x')^2+(y-y')^2+(z-z')^2\right)^{\frac{1}{2}} \tag{C.4}$$

$$= \frac{1}{2}\left((x-x')^2+(y-y')^2+(z-z')^2\right)^{-\frac{1}{2}} \cdot \frac{\partial}{\partial x}(x-x')^2 \tag{C.5}$$

$$= \frac{1}{2}\left((x-x')^2+(y-y')^2+(z-z')^2\right)^{-\frac{1}{2}} \cdot 2(x-x') \tag{C.6}$$

$$= \left((x-x')^2+(y-y')^2+(z-z')^2\right)^{-\frac{1}{2}}(x-x') \tag{C.7}$$

$$= \frac{(x-x')}{\left((x-x')^2+(y-y')^2+(z-z')^2\right)^{\frac{1}{2}}} \tag{C.8}$$

$$= \frac{r_x}{r} \tag{C.9}$$

$$\nabla r = \left(\frac{r_x}{r}, \frac{r_y}{r}, \frac{r_z}{r}\right) \tag{C.10}$$

$$= \frac{(r_x, r_y, r_z)}{r} \tag{C.11}$$

$$= \frac{\mathbf{r}}{r} = \hat{\mathbf{r}} \tag{C.12}$$

■

## C.2 The gradient of $\frac{1}{r}$

*Given a displacement vector* $\mathbf{r}$ *with magnitude* $r = |\mathbf{x} - \mathbf{x}'|$*, then*

*Proof.*

$$\nabla\left(\frac{1}{r}\right) = -\frac{\mathbf{r}}{r^3} \tag{C.13}$$

$$\nabla\left(\frac{1}{r}\right)_x = \left(\frac{\partial}{\partial x}\right)\frac{1}{r} = \left(\frac{\partial}{\partial x}\right)\frac{1}{|\mathbf{x}-\mathbf{x}'|} \tag{C.14}$$

$$= \left(\frac{\partial}{\partial x}\right)\frac{1}{\sqrt{\left((x-x')^2+(y-y')^2+(z-z')^2\right)}} \tag{C.15}$$

$$= \left(\frac{\partial}{\partial x}\right)\left((x-x')^2+(y-y')^2+(z-z')^2\right)^{-\frac{1}{2}} \tag{C.16}$$

$$= -\frac{1}{2}\bigg((x-x')^2 + (y-y')^2 + (z-z')^2\bigg)^{-\frac{3}{2}} \cdot \frac{\partial}{\partial x}(x-x')^2 \tag{C.17}$$

$$= -\frac{1}{2}\bigg((x-x')^2 + (y-y')^2 + (z-z')^2\bigg)^{-\frac{3}{2}} \cdot 2(x-x') \tag{C.18}$$

$$= -\bigg((x-x')^2 + (y-y')^2 + (z-z')^2\bigg)^{-\frac{3}{2}}(x-x') \tag{C.19}$$

$$= -\Bigg(\bigg((x-x')^2 + (y-y')^2 + (z-z')^2\bigg)^{-\frac{1}{2}}\Bigg)^3 (x-x') \tag{C.20}$$

$$= -\bigg(\frac{1}{r}\bigg)^3 (x-x') = -\frac{1}{r^3}\mathbf{r}_x \tag{C.21}$$

$$\nabla\bigg(\frac{1}{r}\bigg) = -\frac{\mathbf{r}}{r^3} \tag{C.22}$$

■

# D Table of Theorems

| Description | GCED | Theorem | Left hand side |
|---|---|---|---|
| Magnetic vector field | **G**[2.3] | 1 | $\mathbf{B}_T(\mathbf{x})$ |
| Magnetic scalar field | **G**[2.8] | 2 | $B_L(\mathbf{x})$ |
| Lorenz condition | **G**[2.14] | 3 | $\nabla\big(B_L+\epsilon_0\mu_0 B_\Phi\big)$ |
| Phi power density | **G**[3.29] | 4 | $-\rho B_\Phi$ |
| Phi force density | **G**[3.30] | 5 | $\rho\mathbf{E}_\Phi$ |
| LEM power density | **G**[3.31] | 6 | $-\mathbf{E}_L\cdot\mathbf{J}_l$ |
| LEM force density | **G**[3.32] | 7 | $B_L\mathbf{J}_l$ |
| TEM power density | **G**[3.33] | 8 | $-\mathbf{E}_T\times\mathbf{J}_t$ |
| TEM force density | **G**[3.34] | 9 | $\mathbf{B}_T\times\mathbf{J}_t$ |
| Superimposed power density | **G**[3.41] | 10 | $-\mathbf{E}\cdot\mathbf{J}-c^2B\rho$ |
| Superimposed force density | **G**[3.42] | 11 | $\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}_T+(\frac{\epsilon_0}{\lambda_0}\mathbf{J}_l+\mathbf{J}_t)B^*$ |
| Superimposed power density | **G**[3.43] | 12 | $-\mathbf{E}\cdot\mathbf{J}-c^2B\rho$ |
| Superimposed force density | **G**[3.44] | 13 | $\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}_T+\mathbf{J}B^*$ |
| Magnetostatic power density | **G**[3.46] | 14 | $-\mathbf{E}_\Phi\cdot\mathbf{J}$ |
| Magnetostatic force density | **G**[3.47] | 15 | $\rho\mathbf{E}_\Phi+\mathbf{J}\times\mathbf{B}_T+(1-\frac{c^2}{a^2})\mathbf{J}B_L$ |
| Vlaenderen's scalar Phi far field | **G**[3.53] | 16 | $B_\Phi(\mathbf{x},\mathrm{t})$ |
| Scalar magnetic far field | **G**[3.54] | 17 | $B_L(\mathbf{x},\mathrm{t})$ |
| Magnetic vector far field | **G**[3.55] | 18 | $\mathbf{B}_T(\mathbf{x},\mathrm{t})$ |
| Electric vector far field | **G**[3.56] | 19 | $\mathbf{E}(\mathbf{x},\mathrm{t})$ |

# References

[1] Koen van Vlaenderen. *General Classical Electrodynamics.* Horizon Research Publishing (HRPUB), 2016.

[2] David J. Griffiths. *Introduction to Electrodynamics.* Fourth Edition, Pearson Education Limited, 2014.

[3] Gennady V. Nikolaev.*Electrodynamics Physical Vacuum.* Tomsk Polytechnical University, 2004.